Department of Mathematics, The Chinese University of Hong Kong
Geometric Analysis Working Seminar 2015/MATH4400C

Cassandra Lee, supervised by Professor Martin Li
Text: Liviu Nicolaescu, *An Invitation To Morse Theory*

December 1, 2015: Section 2.7 Min-max theory, p. 87 – 99.

Our objective now is to extract information about critical points using topological means. Unless otherwise indicated, in this set of notes, we let $f : M \to \mathbb{R}$ be a smooth function on a compact, connected manifold M without boundary equipped with a Riemannian metric g. (If M is not compact, the principles to be presented below involve more technicalities.)

If f has critical points, Morse theory has shown us that we can estimate how many critical points f has at least. In particular, for a Morse function – all its critical points are nondegenerate by definition – the lower bound for the number of critical points is the sum of the Betti numbers of M. (This follows from the Morse inequalities in Section 2.3, esp. p.59. We omit the discussion here. Wikipedia "Morse Theory" has a less formal discussion.) How about smooth functions with degenerate critical points? Can we estimate nontrivially the least number of critical points?

Trivially, a smooth function on a compact manifold M (immersed in $\mathbb{R}^{n+1}$ in which n is the smallest dimension of Euclidean space M could be in) has at least two critical points: a maximum and a minimum. There couldn't be more or fewer because, naively speaking, the height function applied to an n-sphere also has two critical points. Reeb showed that if f only has nondegenerate maxima and minima as critical points, then M must be homeomorphic to an n-sphere. This generalization omits all information about other types of critical points, such as saddle points, which **min-max theory** can recover.

Set-up: Denote $M^c := \{f \leq c\}$ where $c \in \mathbb{R}$. We shall lay some groundwork on the min-max principle and its applications in locating nontrivial critical points, and build everything up to the nontrivial estimate of critical points we've been seeking.

Definition 2.50. A collection of *min-max data* for f is a pair $(\mathcal{H}, \mathcal{S})$ satisfying the following criteria:

(a) $\mathcal{H} \subset \text{Homeo}(M)$: $\forall a \in M \backslash \mathbf{\Delta}_f, \exists \varepsilon > 0, \exists h \in \mathcal{H}$ such that $h(M^{a+\varepsilon}) \subset M^{a-\varepsilon}$, and

(b) $\mathcal{S} \subset \mathcal{P}(M)$: $h(S) \in \mathcal{S}, \forall h \in \mathcal{H}, \forall S \in \mathcal{S}$. ($\mathcal{S}$ is invariant under $\mathcal{H}$.)

Theorem 2.51 (Min-max principle: the key existence result here). If $(\mathcal{H}, \mathcal{S})$ is a collection of min-max data for the smooth function $f : M \to \mathbb{R}$, then the following real number (real-valued function of $(\mathcal{H}, \mathcal{S})$) is a critical value of f:

$$c = c(\mathcal{H}, \mathcal{S}) := \inf_{S \in \mathcal{S}} \sup_{x \in S} f(x).$$

Proof: Basically we're copying the statements above and using very elementary analysis to reach a contradiction.

Suppose c is a regular value. Then $\exists \varepsilon > 0, \exists h \in \mathcal{H}$ such that $h(M^{c+\varepsilon}) \subset M^{c-\varepsilon}$. Now, c being an infimum, $\exists S \in \mathcal{S}$ such that

$$\sup_{x \in S} f(x) < c + \varepsilon \Longrightarrow S \subset M^{c+\varepsilon}.$$

Therefore, $S' := h(S) \in \mathcal{S}$ and $S' = h(S) \subset M^{c-\varepsilon}$

$$\Rightarrow \sup_{y \in S'} f(y) \leq c - \varepsilon \Rightarrow \inf_{S' \in \mathcal{S}} \sup_{y \in S'} f(y) \leq c - \varepsilon$$

but that leads to the contradition

$$c = \inf_{S \in \mathcal{S}} \sup_{x \in S} f(x) \leq c - \varepsilon.$$

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