Department of Mathematics, The Chinese University of Hong Kong
Geometric Analysis Working Seminar 2015/MATH4400C

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Text: Liviu Nicolaescu, *An Invitation To Morse Theory*

October 13, 2015: Section 1.1, p. 5 – 17 up to just before **Section 1.2**.

But we still have not addressed the heart of Morse theory yet. Up till this point, we are only dealing with upgraded advanced calculus. To move on, recall the second derivative test in high school. If $x_0 \in \mathbb{R}$ is a critical point of a second-differentiable function $f : \mathbb{R} \to \mathbb{R}$, x_0 is a

$$\begin{cases} \text{maximum point if } f''(x_0) < 0, \\ \text{minimum point if } f''(x_0) > 0. \end{cases}$$

How about the higher-dimensional analogue? It's the Hessian, of course: eigenvalues all negative correspond to the critical point being a maximum point, all positive a minimum point, and eigenvalues of opposite signs indicate a saddle point.

Yet our basic undergraduate curriculum does not adequately cover what the "second derivative" is for $n \geq 3$. We need a more general notion of the Hessian. Note that we are concerned with the Hessian's eigenvalues, which in turn depend on the underlying basis functions, the *vector fields*. In Euclidean space, given a smooth object $M \subset \mathbb{R}^n$, we know that a vector field on M is a map from M to $\mathbb{R}^n$, assigning to each point an n-dimensional vector.

This naive notion, however, does not capture the nature of the output vectors – they are tangent vectors to M. Therefore, to be more precise, given a manifold M, we say that a *vector field* on M is a map from M to the collection of all M's tangent spaces – its tangent bundle TM.

Let $p_0 \in M$ be a critical point of a smooth function $f : M \to \mathbb{R}$. The text uses a certain rather convenient notation: $Pf := \mathrm{d}f(P)$ where $P : M \to TM$ is a vector field. $\mathrm{d}f$ is the differential map of f and its input is the vector field P itself. With these, we would like to define the *Hessian* of f at p_0 to be the map

$$H_{f,p_0} : T_{p_0}M \times T_{p_0}M \to \mathbb{R}, (X(p_0), Y(p_0)) \mapsto (XYf)(p_0)$$

. A problem remains. Is this definition well-defined? Will different vector field extensions at the critical point p_0 lead to different Hessian functions? Let's check:

Lemma 1.6: Suppose $f : M \to \mathbb{R}$ is a smooth function and $p_0 \in M$ is a critical point of f. Then for every choice of vector fields X, X', Y and Y' on M such that $X(p_0) = X'(p_0), Y(p_0) = Y'(p_0)$, we have $(XYf)(p_0) = (X'Y')f(p_0) = (YXf)(p_0)$. That a point is a critical point is invariant under a change of vector fields.

Proof: This proof requires knowledge of the *Lie bracket*.[1]. Simply put, a Lie

[1] Suggested reading: *Introduction to Smooth Manifolds*, John M. Lee, p.329, "Lie Brackets"

bracket is a vector field with two smooth vector fields P, Q on a smooth manifold M and a smooth function $f : M \to \mathbb{R}$ as inputs, defined by $[P, Q]f := PQf - QPf$. It "measures" the extent to which the derivatives of f in X and Y do not commute.

Note first that $\mathrm{d}f(XY) - \mathrm{d}f(YX) = (XY - YX)f(p_0) = ([X, Y]f)(p_0) = \mathrm{d}f([X, Y])(p_0) = 0$. The last equality follows from the fact that f is smooth and hence twice-differentiable, and so its directional derivatives along X and Y commute: $\mathrm{d}f(XY) = \mathrm{d}f(YX)$. If X, Y are both defined on a one-dimensional manifold and x, y are their respective coordinates, $\mathrm{d}f(XY) = \mathrm{d}f(YX)$ means $\frac{\partial}{\partial x}\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\frac{\partial f}{\partial x}$.

Since $(X - X')(p_0) = 0$ and the differential map is linear, mapping the zero tangent vector in the domain to the zero tangent vector in the range, with $C^\infty(M)$ denoting the set of smooth maps from M to itself, we deduce that

$$(X - X')g(p_0) = \mathrm{d}g(X - X')(p_0) = \mathrm{d}g(0) = 0, \forall g \in C^\infty(M).$$

Hence $(X - X')Yf(p_0) = 0 \implies (XYf)(p_0) = (X'Yf)(p_0)$. As we can swap X with X', we also swap Y and Y', replace Y with Y', and then have the resultant vector fields exchange places: $(X'Yf)(p_0) = (YX'f)(p_0) = (Y'X'f)(p_0) = (X'Y'f)(p_0)$.■

Thanks to this lemma, the Hessian H_{f,p_0} defined above is bilinear and symmetric. It's a symmetric bilinear form with a square matrix for its matrix representation. ☺

Definition 1.7: *(Nondegeneracy and Morse function)* A critical point $p_0 \in M$ of a smooth function $f : M \to \mathbb{R}$ is called *nondegenerate* if its Hessian is nondegenerate. That means $H_{f,p_0}(X, Y) = 0$ for all $Y \in T_{p_0}M$ if and only if $X = 0$. A smooth function is called a *Morse function* if all its critical points are nondegenerate.

So far we're only playing with abstract ideas. Can we see them as coordinates of n-dimensional space? Yes. The text follows the formal notations of standard geometry texts. We introduce them painlessly still using things defined above:

If we consider the Euclidean n-dimensional space surrounding p_0, we could specify coordinates $(x^1, x^2, ..., x^n)$ such as (x, y, z) for $\mathbb{R}^3$ for each point in that n-space. Superscripts go with scalar coefficients; exponentiation, the "default" idea of a superscript, simply requires an extra bracket followed by the exponent.

With these, we choose $(x^1, x^2, ..., x^n)$ such that $x^k(p_0) = 0, k = 1, ..., n$, which means that p_0 is the "origin" of the n-dimensional neighbourhood of p_0. Since "e" has many other uses, we use the geometer's notation $\{\frac{\partial}{\partial x^k}\}_{k=1}^n$ that corresponds to the linear algebraic standard basis $\{e_k\}_{k=1}^n$. Then $X = \sum_i X^i \frac{\partial}{\partial x^i}$ and $Y = \sum_j Y^j \frac{\partial}{\partial x^j}$, where X^i, Y^j are coefficients in the coordinate system we have specified near p_0.

Furthermore, $H_{f,p_0}(X, Y) = \sum_{i,j} h_{ij} X^i Y^j, h_{ij} = \frac{\partial}{\partial x^i}\frac{\partial}{\partial x^j} f(p_0)$. Notice that the same notation stands for derivatives and for the standard basis of vector fields because we are considering tangent vectors (which are derivatives in higher dimensions) here. The critical point is nondegenerate if and only if $\det(h_{ij} \neq 0)$.

The Hessian also determines the function $H_{f,p_0}(x) = \sum_{i,j} h_{ij} x^i x^j$ defined in a neighborhood of p_0. This is the second-order term in the Taylor expansion of f at p_0: $f(x) = f(p_0) + \frac{1}{2} H_{f,p_0}(x) + \mathcal{O}(3)$.

How to determine the type of nondegenerate critical points

The text mentions the following "classical fact of linear algebra" (modified slightly):

If V is a real vector space of finite dimension n and $b : V \times V \to \mathbb{R}$ is a symmetric, bilinear nondegenerate map, then there exists some basis $(e_1, ..., e_n)$ such that for any $v = \sum_i v^i e_i$, we have

$$b(v,v) = -\sum_{i=1}^{\lambda} |v^i|^2 + \sum_{j=\lambda+1}^{n} |v^j|^2. \tag{1}$$

At first sight, it doesn't look like anything we have known. However, it turns out to be Sylvester's Law of Inertia. Compare:

Let H be a symmetric bilinear form on a finite-dimensional real vector space V. Then the number of positive diagonal entires and the number of negative diagonal entries in any diagonal matrix representation of H are independent of the diagonal representation.[2]

Then by a certain scaling on the basis elements, b's matrix representation becomes a diagonal matrix with $1, -1, 0$ as its only entries, just like the "fact" in the text. Note that, I_r being the $r \times r$ identity matrix, we can rewrite equation (1) as

$$b(v,v) = v^T \begin{pmatrix} -I_\lambda & O \\ O & I_{n-\lambda} \end{pmatrix} v. \tag{2}$$

Let $B = \left(\begin{smallmatrix} -I_\lambda & O \\ O & I_{n-\lambda} \end{smallmatrix}\right)$. Conventionally, the *index* of b is the number of positive diagonal entries in B, but in the study of Morse theory, we change "positive" to "negative" – reminding us of *Liapunov's second method* used to classify stability of critical points (no ambiguity here: regard asymptotically stable critical points as minimum points, and unstable ones saddle and maximum points) in studying nonlinear ordinary differential equations. The technicalities aside, given a function F and a critical point p_0 of it, $\dot{F}$ being negative definite at a neighbourhood containing p_0 corresponds to a minimum; positive definite, a maximum. To measure anything in between, we change the sign of each $|\text{coefficient}|^2$ and observe when b first becomes negative – then the smallest λ that makes this happen is the index, defined as "the largest integer l such that there exists an l-dimensional subspace V_- of V with the property that the restriction of b to V_- is negative definite."

[2] Taken from Friedberg, Insel, Spence *Linear Algebra*, P.443.